

Unified Parametrization for Quark and Lepton Mixing Angles

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Abstract

We propose a new parametrization for the quark and lepton mixing matrices: the two 12-mixing angles (the Cabibbo angle and the angle responsible for solar neutrino oscillations) are at zeroth order $\pi/12$ and $\pi/5$, respectively. The resulting 12-elements in the CKM and PMNS matrices, V_{us} and U_{e2} , are in this order irrational but simple algebraic numbers. We note that the cosine of $\pi/5$ is the golden ratio divided by two. The difference between $\pi/5$ and the observed best-fit value of solar neutrino mixing is of the same order as the difference between the observed value and the one for tri-bimaximal mixing. In order to reproduce the central values of current fits, corrections to the zeroth order expressions are necessary. They are small and of the same order and sign for quarks and leptons. We parametrize the perturbations to the CKM and PMNS matrices in a “triminimal” way, i.e., with three small rotations in an order corresponding to the order of the rotations in the PDG-description of mixing matrices.

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Quark and lepton mixing can successfully be described in the Standard Model of elementary particle physics. For each sector a unitary mixing matrix connects mass and flavor states, and can be parametrized as [1]

$$V, U = R_{23}(\theta_{23}^{q,\ell}) R_{13}^{q,\ell}(\theta_{13}^{q,\ell}; \delta^{q,\ell}) R_{12}(\theta_{12}^{q,\ell}) \\ = \begin{pmatrix} c_{12}^{q,\ell} c_{13}^{q,\ell} & s_{12}^{q,\ell} c_{13}^{q,\ell} & s_{13}^{q,\ell} e^{-i\delta^{q,\ell}} \\ -s_{12}^{q,\ell} c_{23}^{q,\ell} - c_{12}^{q,\ell} s_{23}^{q,\ell} s_{13}^{q,\ell} e^{i\delta^{q,\ell}} & c_{12}^{q,\ell} c_{23}^{q,\ell} - s_{12}^{q,\ell} s_{23}^{q,\ell} s_{13}^{q,\ell} e^{i\delta^{q,\ell}} & s_{23}^{q,\ell} c_{13}^{q,\ell} \\ s_{12}^{q,\ell} s_{23}^{q,\ell} - c_{12}^{q,\ell} c_{23}^{q,\ell} s_{13}^{q,\ell} e^{i\delta^{q,\ell}} & -c_{12}^{q,\ell} s_{23}^{q,\ell} - s_{12}^{q,\ell} c_{23}^{q,\ell} s_{13}^{q,\ell} e^{i\delta^{q,\ell}} & c_{23}^{q,\ell} c_{13}^{q,\ell} \end{pmatrix} P^{q,\ell}. \quad (1)$$

Here V is the Cabibbo-Kobayashi-Maskawa matrix (CKM) matrix containing the mixing angles θ_{ij}^q and U is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix containing the mixing angles θ_{ij}^ℓ . As usual, $c_{ij}^{q,\ell} = \cos \theta_{ij}^{q,\ell}$ and $s_{ij}^{q,\ell} = \sin \theta_{ij}^{q,\ell}$. There is also a diagonal phase matrix, which is trivial in the quark sector ($P^q = \mathbb{1}$), and contains the Majorana phases in the lepton sector: $P^\ell = \text{diag}(1, e^{i\alpha}, e^{i\beta})$. As indicated, the above mixing matrices are products of rotations, e.g.

$$R_{23}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}, \quad R_{13}(\theta; \delta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin \theta e^{i\delta} & 0 & \cos \theta \end{pmatrix}.$$

Impressive progress has been achieved in determining the parameters of U and V . The current knowledge for the quark sector can be summarized at 1σ as [1]

$$\begin{aligned} \sin \theta_{12}^q &= 0.2257 \pm 0.0010, \\ \sin \theta_{23}^q &= 0.0415^{+0.0010}_{-0.0011}, \\ \sin \theta_{13}^q &= 0.00359 \pm 0.00016. \end{aligned} \quad (2)$$

The CP phase lies in the range (see also e.g. [2]) $\delta^q = (68.85^{+3.04}_{-5.42})^\circ$. Regarding the leptons, we have at 1, 2 and 3σ [3]

$$\begin{aligned} \sin \theta_{12}^\ell &= 0.559^{+0.017, 0.035, 0.054}_{-0.016, 0.031, 0.046}, \\ \sin \theta_{23}^\ell &= 0.683^{+0.052, 0.093, 0.120}_{-0.044, 0.078, 0.107}, \\ \sin \theta_{13}^\ell &= 0.126^{+0.035, 0.063, 0.088}_{-0.049, 0.126, 0.126}. \end{aligned} \quad (3)$$

The hint for non-zero θ_{13}^ℓ [3] (see also [4]) is rather weak, surviving only 1.6σ . There is no information on leptonic CP violation.

The precision era which also neutrino physics has entered recently allows in particular to parametrize the PMNS matrix with reasonable accuracy. Trying to parametrize the mixing matrices [5–15] might be helpful phenomenologically as well as could hint towards a structure underlying the observed mixing patterns. There are three desiderata for convenient parametrizations:

- a) fast convergence, i.e., the zeroth order expression should be close to the observed values;
- b) at zeroth order the mixing matrix entries should be simple numbers;
- c) similar parametrizations should be used for both quarks and leptons.

The latter implies usually that very different zeroth order forms of U and V are needed. For instance, one could use the unit matrix for the quarks and tri-bimaximal [16] mixing (or bimaximal [17]) for the leptons. In this letter we will propose to use in the zeroth order matrices angles which are fractions of π with an integer number. This is of course common practice for atmospheric mixing, $(\theta_{23}^\ell)^0 = \pi/4$, and will be used here for the 12-rotations of the quarks $((\theta_{12}^q)^0 = \pi/12)$ and leptons $((\theta_{12}^\ell)^0 = \pi/5)$ as well. The value $(\theta_{12}^\ell)^0 = \pi/5$ is in fact within the allowed 2σ range. Our alternative Ansatz brings the zeroth order forms of U and V on somewhat equal footing and can be used for the quark and lepton sectors simultaneously. Of course, the angles themselves are not physical and our phenomenological Ansatz relies on the particular parametrization of Eq. (1). Nevertheless, the sines and cosines of the angles are almost the mixing matrix elements, which are indeed physical quantities¹. With the angles chosen here, the sines and cosines turn out to be simple and concise irrational numbers. This is reminiscent of bimaximal or tri-bimaximal mixing. In addition, comparing our proposal with current best-fit values reveals that the required perturbation parameters are small and for the lepton sector of the same order as the required perturbation parameters for deviations from tri-bimaximal mixing. They are furthermore of the same order and sign for both quarks and leptons, at least for the 12-sector. We have thus met all three desiderata given above.

Having a zeroth order form of the mixing matrix implies that in general (ignoring the Majorana phases of the lepton sector) four small parameters $\epsilon_{ij}^{q,\ell}$ have to be introduced, corresponding to the four parameters describing all mixing phenomena. If the four parameters are introduced in terms of (small angle) rotations, it is most straightforward to choose the order of rotations such that each small parameter $\epsilon_{ij}^{q,\ell}$ is responsible for the deviation of (and only of) $\theta_{ij}^{q,\ell}$ from its initial value. This “triminimal” parametrization [13] has been applied for the tri-bimaximal [16] mixing scheme (TBM). The latter corresponds to $(\theta_{23}^\ell)^0 = \pi/4$, $(\theta_{13}^\ell)^0 = 0$ and $(\theta_{12}^\ell)^0 = \theta_{\text{TBM}}$, or $U^0 = R_{23}(\pi/4) R_{12}(\theta_{\text{TBM}})$, where $\sin^2 \theta_{\text{TBM}} = \frac{1}{3}$. One then parametrizes the PMNS matrix triminimally as [13]

$$U = R_{23}(\pi/4) R_{23}(\epsilon_{23}^\ell) R_{13}(\epsilon_{13}^\ell; \delta^\ell) R_{12}(\epsilon_{12}^\ell) R_{12}(\theta_{\text{TBM}}). \quad (4)$$

The order of small rotations in between the rotations with the large angles $\pi/4$ and θ_{TBM} is the same as in the PDG-description of a mixing matrix. It is then easy to see that each ϵ_{ij}^ℓ describes the deviation of (and only of) θ_{ij}^ℓ from $(\theta_{ij}^\ell)^0$. Moreover, the introduced CP phase appears exactly where it appears in Eq. (1). Note that a triminimal parametrization

¹ Ref. [15] has recently proposed to use the value $\sin(\theta_{12}^q)^0 = \frac{\sqrt{2}-1}{\sqrt{6}}$, or $(\theta_{12}^q)^0 \simeq \pi/18.49$, as the zeroth order expression for the Cabibbo angle.

is manifestly unitary. If it turns out that one of the deviations from tri-bimaximal mixing is sizable, this parametrization can treat that case more accurately. Regarding the quark sector, the hierarchy in the CKM angles implies to start with only a non-zero 12-rotation and then introduce from the left in a triminimal way three small rotations in the order of the PDG parametrization. This has recently been proposed in Ref. [15].

Let us start by considering the quark sector. The goal is to find a simple initial mixing angle $(\theta_{12}^q)^0$, being a fraction of π with an integer number, and which could be used as starting point for an expansion. In addition it should yield a simple number for the sine and cosine. This leads to the choice²

$$(\theta_{12}^q)^0 = \frac{\pi}{12} \Rightarrow \sin(\theta_{12}^q)^0 = \frac{\sqrt{3}-1}{2\sqrt{2}} = 0.2588, \quad (5)$$

such that at zeroth order the CKM matrix is

$$V^0 = \begin{pmatrix} \frac{\sqrt{3}+1}{2\sqrt{2}} & \frac{\sqrt{3}-1}{2\sqrt{2}} & 0 \\ \frac{1-\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}+1}{2\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.9659 & 0.2588 & 0 \\ -0.2588 & 0.9659 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

In the spirit of triminimality [13], we can then describe the small deviations of this matrix with

$$V = R_{23}(\epsilon_{23}^q) R_{13}(\epsilon_{13}^q; \delta^q) R_{12}(\epsilon_{12}^q) V^0. \quad (7)$$

As mentioned above, a triminimal parametrization has the obvious advantage that each small parameter ϵ_{ij}^q is responsible for the deviation of θ_{ij}^q from its initial value. In this case ϵ_{23}^q is θ_{23}^q , and $V_{ub} = \sin \epsilon_{13}^q e^{-i\delta^q}$. The allowed ranges of $\epsilon_{13,23}^q$ are nothing but the allowed ranges of the parameters $\theta_{13,23}^q$ given in Eq. (2). In addition, δ^q is directly interpretable as the CP phase in the usual PDG parametrization. Expanding the CKM matrix to first order gives

$$V \simeq V^0 + \epsilon_{12}^q \begin{pmatrix} \frac{1-\sqrt{3}}{2\sqrt{2}} & \frac{1+\sqrt{3}}{2\sqrt{2}} & 0 \\ -\frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \epsilon_{13}^q \begin{pmatrix} 0 & 0 & e^{-i\delta^q} \\ 0 & 0 & 0 \\ -\frac{1+\sqrt{3}}{2\sqrt{2}} e^{i\delta^q} & \frac{1-\sqrt{3}}{2\sqrt{2}} e^{i\delta^q} & 0 \end{pmatrix} \\ + \epsilon_{23}^q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1-\sqrt{3}}{2\sqrt{2}} & -\frac{1+\sqrt{3}}{2\sqrt{2}} & 0 \end{pmatrix}. \quad (8)$$

² Actually, a fraction of π which is slightly closer to the measured value of the Cabibbo angle is $(\theta_{12}^q)^0 = \frac{\pi}{14} \simeq 0.2244$. We note that $V_{us} = \sin \frac{\pi}{14}$ has been obtained from a flavor symmetry in Refs. [18]. It was shown that if a dihedral group D_n (generated by the generators A and B) is broken by Higgs vevs to different Z_2 subgroups (BA^{m_u} in the up-quark sector, BA^{m_d} for the down-quarks), then $|V_{us}| = |\cos \pi(m_u - m_d)j/n|$ is obtained. Here j is the index of the representation under which the quarks transform. In case $j = 1$, $m_u = 3$, $m_d = 0$ and $n = 7$ it follows that $|V_{us}| = \cos \frac{3\pi}{7} = \sin \frac{\pi}{14}$. Similar considerations can be performed for other mixing angles. Anyway, here we restrict ourselves to pure phenomenology.

The sine of the 12-mixing angle is given by

$$\sin \theta_{12}^q = \frac{1}{2} \sqrt{2 - \sqrt{3} \cos 2\epsilon_{12}^q + \sin 2\epsilon_{12}^q} \simeq \frac{\sqrt{3}-1}{2\sqrt{2}} \left(1 + (2 + \sqrt{3}) \epsilon_{12}^q \right). \quad (9)$$

Note that the last expression is equivalent to $\sin \theta_{12}^q \simeq \sin(\theta_{12}^q)^0 + \epsilon_{12}^q \cos(\theta_{12}^q)^0$. Numerically we have $\sin \theta_{12}^q \simeq 0.2588 + 0.9659 \epsilon_{12}^q$, so that ϵ_{12}^q can be almost directly identified with the deviation of the sine of Cabibbo angle from $\frac{\sqrt{3}-1}{2\sqrt{2}}$. In order to bring $\sin \theta_{12}^q$ in the observed range given in Eq. (2) one requires

$$\epsilon_{12}^q = -0.0341 \pm 0.0010. \quad (10)$$

There is a hierarchy implied for the small parameters, namely $(\epsilon_{12}^q)^2 \sim (\epsilon_{23}^q)^2 \sim \epsilon_{13}^q$. The Jarlskog invariant $J_{\text{CP}}^q = \text{Im}(V_{us} V_{cs}^* V_{ub}^* V_{cb})$ is

$$J_{\text{CP}}^q = \frac{1}{16} \cos \epsilon_{13}^q \sin 2\epsilon_{13}^q \sin 2\epsilon_{23}^q (\cos 2\epsilon_{12}^q + \sqrt{3} \sin 2\epsilon_{12}^q) \sin \delta^q. \quad (11)$$

In the limit of $\delta^q = \pi/3$ and $\theta_{12}^q = \pi/12$ we have $J_{\text{CP}}^q = \frac{\sqrt{3}}{32} \sin 2\theta_{23}^q \sin 2\theta_{13}^q \cos \theta_{13}^q \simeq \frac{\sqrt{3}}{8} \theta_{23}^q \theta_{13}^q$.

In principle one could also start with non-zero zeroth order expressions for $\sin \theta_{23,13}^q$ in which the angles are also written as π/n , for instance $\theta_{23}^q = \frac{\pi}{76} \simeq 0.0413$ and $\theta_{13}^q = \frac{\pi}{875} \simeq 0.00359$. Since $\theta_{23,13}^q$ are very small, this is not necessary. Note however that $\delta^q = \pi/3$ is a good approximation.

Turning to the lepton sector, it is trivial to note that atmospheric neutrino mixing can be described with $(\theta_{23}^\ell)^0 = \pi/4$ and that $(\theta_{13}^\ell)^0 = 0$. In analogy to the discussion for the quark sector, we propose to use as the zeroth order expression for the (solar) 12-rotation

$$(\theta_{12}^\ell)^0 = \frac{\pi}{5} \Rightarrow \sin^2(\theta_{12}^\ell)^0 = \frac{5 - \sqrt{5}}{8} \simeq 0.345. \quad (12)$$

As can be seen from Eq. (3), this value is within the 2σ range of the oscillation parameters. The corresponding 12-rotation is

$$R_{12}(\pi/5) = \frac{1}{4} \begin{pmatrix} 1 + \sqrt{5} & \sqrt{2}\sqrt{5 - \sqrt{5}} & 0 \\ -\sqrt{2}\sqrt{5 - \sqrt{5}} & 1 + \sqrt{5} & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0.809 & 0.588 & 0 \\ -0.588 & 0.809 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (13)$$

Note that $\cos \frac{\pi}{5} = \varphi/2$, where $\varphi = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio³. Consequently, $\sin^2 \frac{\pi}{5} = \frac{1}{4}(3 - \varphi)$, $\sin^2 2\frac{\pi}{5} = \frac{\sqrt{5}}{4} \varphi$, etc. The zeroth order PMNS matrix is (omitting the Majorana

³Ref. [19] has proposed to identify the cotangent of θ_{12}^ℓ with the golden ratio. That this value is close to the best-fit point has been also observed in Ref. [20]. We note that our choice of $\cos \theta_{12}^\ell = \varphi/2$ is closer to the best-fit point.

phases)

$$U^0 = \begin{pmatrix} \frac{1+\sqrt{5}}{4} & \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}} & 0 \\ -\frac{\sqrt{5-\sqrt{5}}}{4} & \frac{1+\sqrt{5}}{4\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sqrt{5-\sqrt{5}}}{4} & \frac{1+\sqrt{5}}{4\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0.809 & 0.588 & 0 \\ -0.416 & 0.572 & -0.707 \\ -0.416 & 0.572 & 0.707 \end{pmatrix}. \quad (14)$$

The triminimal description of the PMNS matrix is then

$$U = R_{23}(-\pi/4) R_{23}(\epsilon_{23}^\ell) R_{13}(\epsilon_{13}^\ell; \delta^\ell) R_{12}(\epsilon_{12}^\ell) R_{12}(\pi/5). \quad (15)$$

We find from this expression that $U_{e3} = \sin \epsilon_{13}^\ell e^{-i\delta^\ell}$. Expanding the PMNS matrix in terms of the small parameters gives

$$U \simeq U^0 + \epsilon_{12}^\ell \begin{pmatrix} -\frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}} & \frac{1+\sqrt{5}}{4} & 0 \\ -\frac{1+\sqrt{5}}{4\sqrt{2}} & -\frac{\sqrt{5-\sqrt{5}}}{4} & 0 \\ -\frac{1+\sqrt{5}}{4\sqrt{2}} & -\frac{\sqrt{5-\sqrt{5}}}{4} & 0 \end{pmatrix} + \epsilon_{13}^\ell \begin{pmatrix} 0 & 0 & e^{-i\delta^\ell} \\ \frac{1+\sqrt{5}}{4\sqrt{2}} e^{i\delta^\ell} & \frac{\sqrt{5-\sqrt{5}}}{4} e^{i\delta^\ell} & 0 \\ -\frac{1+\sqrt{5}}{4\sqrt{2}} e^{i\delta^\ell} & -\frac{\sqrt{5-\sqrt{5}}}{4} e^{i\delta^\ell} & 0 \end{pmatrix} \\ + \epsilon_{23}^\ell \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\sqrt{5-\sqrt{5}}}{4} & \frac{1+\sqrt{5}}{4\sqrt{2}} & \sqrt{\frac{1}{2}} \\ \frac{\sqrt{5-\sqrt{5}}}{4} & -\frac{1+\sqrt{5}}{4\sqrt{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \quad (16)$$

Moreover, the other trigonometric functions of the mixing angles are

$$\sin^2 \theta_{23}^\ell = \frac{1}{2} (1 - \sin 2\epsilon_{23}^\ell) \simeq \frac{1}{2} - \epsilon_{23}^\ell \quad (17)$$

and

$$\sin^2 \theta_{12}^\ell = \frac{1}{16} \left(\sqrt{2} \sqrt{5-\sqrt{5}} \cos \epsilon_{12}^\ell + (1+\sqrt{5}) \sin \epsilon_{12}^\ell \right)^2 \\ \simeq \frac{5-\sqrt{5}}{8} + \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}} \epsilon_{12}^\ell. \quad (18)$$

Numerically, $\sin^2 \theta_{12}^\ell \simeq 0.345 + 0.951 \epsilon_{12}^\ell$, so that ϵ_{12}^ℓ is to a good precision the deviation of $\sin^2 \theta_{12}^\ell$ from $\frac{5-\sqrt{5}}{8}$. In order to lie in the observed 1σ range of $\sin^2 \theta_{12}^\ell$ given in Eq. (3) we require

$$\epsilon_{12}^\ell = -0.036 \pm 0.020. \quad (19)$$

With $\epsilon_{12}^\ell = -0.013$ we would obtain $\sin^2 \theta_{12}^\ell = \frac{1}{3}$. The result of the triminimal perturbation of tri-bimaximal mixing, i.e., using Eq. (4), would be $(\epsilon_{12}^\ell)_{\text{TBM}} = -0.023 \pm 0.020$. This means that the deviations of the tri-bimaximal value $\theta_{\text{TBM}} \simeq 0.615$ and of $\pi/5$ from the best-fit value are of comparable magnitude.

Comparing Eq. (19) with the result for the quark sector in Eq. (10), we find the remarkable result that the best-fit values of the 12-mixing angles are away from $\pi/12$ and $\pi/5$ by the

same (small) amount of roughly⁴ -0.035 . Note however that the error on the leptonic parameter ϵ_{12}^ℓ is much larger than that on ϵ_{12}^q . Nevertheless, this may motivate one to assume a unified Ansatz in what regards the small parameters of triminimality:

$$\epsilon_{ij}^q = \epsilon_{ij}^\ell. \quad (20)$$

This would lead to $\sin^2 \theta_{12}^\ell \simeq 0.313$ and $\sin^2 \theta_{23}^\ell \simeq \frac{1}{2} - |V_{cb}| \simeq 0.458$, which are remarkably close to the best-fit values 0.312 and 0.466 quoted in Eq. (3). The remaining unknown mixing parameter is “predicted” to be $|U_{e3}| = |V_{ub}| \simeq 0.00359$ and there is no chance to relate it to the (weak) hint for non-zero $|U_{e3}|$, or to measure it in currently planned laboratory experiments.

Leaving these speculations aside, we have that δ^ℓ is currently unconstrained, that (at 2σ) $\epsilon_{13}^\ell \leq 0.036$ and that ϵ_{23}^ℓ is to a good precision nothing but the deviation of $\sin^2 \theta_{23}^\ell$ from $\frac{1}{2}$. To lie in the 1σ (2σ) range one finds $(-0.103) - 0.039 \leq \epsilon_{23}^\ell \leq 0.093$ (0.136). To have θ_{12}^ℓ in its allowed 2σ range we require $\epsilon_{12}^\ell = -0.036_{-0.037}^{+0.042}$. Recall that for the quark parameters a hierarchy in the form of $(\epsilon_{12}^q)^2 \sim (\epsilon_{23}^q)^2 \sim \epsilon_{13}^q$ was implied. In the lepton sector the current lack of comparable precision still allows scenarios like $|\epsilon_{12}^\ell| \sim |\epsilon_{23}^\ell| \sim \epsilon_{13}^\ell$ or $|\epsilon_{23}^\ell| \sim (\epsilon_{13}^\ell)^2$. In principle, one, two or even all ϵ_{ij}^ℓ could be zero, without being outside the allowed 2σ ranges.

Finally, the invariant $J_{\text{CP}}^\ell = -\text{Im}\{U_{e2} U_{\mu 2}^* U_{e3} U_{\mu 3}\}$, which governs leptonic CP violation, is found to be

$$\begin{aligned} J_{\text{CP}}^\ell &= \frac{1}{32} \cos 2\epsilon_{23}^\ell \sin 2\epsilon_{13}^\ell \cos \epsilon_{13}^\ell \left((\sqrt{5} - 1) \sin 2\epsilon_{12}^\ell + \sqrt{2} \sqrt{5 + \sqrt{5}} \cos 2\epsilon_{12}^\ell \right) \sin \delta^\ell \\ &\simeq \frac{1}{8} \left(\frac{\sqrt{5 + \sqrt{5}}}{\sqrt{2}} + \epsilon_{12}^\ell (\sqrt{5} - 1) \right) \epsilon_{13}^\ell \sin \delta^\ell. \end{aligned} \quad (21)$$

In case the relation (20) holds we have $J_{\text{CP}}^\ell \simeq 8.3 \cdot 10^{-4} \sin \delta^\ell$. It is tempting to assume in addition that $\delta^\ell = \delta^q$, leading to $J_{\text{CP}}^\ell \simeq 7.2 \cdot 10^{-4}$.

One may wonder what kind of Majorana neutrino mass matrix, which is defined as $m_\nu = U \text{diag}(m_1, m_2, m_3) U^T$, can give rise to $\theta_{23}^\ell = -\pi/4$, $\theta_{13}^\ell = 0$ and $\theta_{12}^\ell = \pi/5$. The required form is

$$m_\nu = \begin{pmatrix} A & B & B \\ \cdot & \frac{1}{2}(A + xB) + E & \frac{1}{2}(A + xB) - E \\ \cdot & \cdot & \frac{1}{2}(A + xB) + E \end{pmatrix}, \quad (22)$$

where $x = 2\sqrt{2} \sqrt{1 - 2/\sqrt{5}} \simeq 0.919$. The neutrino masses are $m_1 = A - Bx\sqrt{5}/2$, $m_2 e^{2i\alpha} = A + \sqrt{2} \sqrt{1 + 2/\sqrt{5}} B$ and $m_3 e^{2i\beta} = 2E$.

Simple formulae can also be obtained for the phase-averaged mixing probabilities for neutrinos with flavor α to end up with flavor β : $\bar{P}_{\alpha\beta} = \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2$. This expression is valid when the neutrino oscillation length $4E/\Delta m^2$ is much smaller than the travelled

⁴This number is in magnitude amusingly close to the ratio of the solar and atmospheric mass-squared differences.

distance, which is fulfilled, e.g., for high energy astrophysical neutrinos. Expanding up to first order in the small parameters ϵ_{ij}^ℓ one finds

$$\begin{aligned} \overline{P} = \frac{1}{16} \left[\begin{pmatrix} 11 - \sqrt{5} & \frac{1}{2}(5 + \sqrt{5}) & \frac{1}{2}(5 + \sqrt{5}) \\ \cdot & \frac{1}{4}(27 - \sqrt{5}) & \frac{1}{4}(27 - \sqrt{5}) \\ \cdot & \cdot & \frac{1}{4}(27 - \sqrt{5}) \end{pmatrix} - \epsilon_{12}^\ell \sqrt{2} \sqrt{5 - \sqrt{5}} \begin{pmatrix} 4 & -2 & -2 \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix} \right. \\ \left. + \left(\cos \delta^\ell \epsilon_{13}^\ell \sqrt{2} \sqrt{5 - \sqrt{5}} - \epsilon_{23}^\ell (5 + \sqrt{5}) \right) \begin{pmatrix} 0 & -1 & 1 \\ \cdot & 1 & 0 \\ \cdot & \cdot & -1 \end{pmatrix} \right]. \end{aligned} \quad (23)$$

As is well-known, the 12-rotation of the lepton sector plays a crucial role in neutrinoless double beta decay, in particular if neutrino masses are inversely hierarchical or quasi-degenerate. In case of an inverted hierarchy ($m_2 \simeq m_1 \gg m_3$) the effective mass $|(m_\nu)_{ee}|$, on which the rate of neutrinoless double beta decay depends quadratically, is

$$|(m_\nu)_{ee}| \simeq \cos^2 \theta_{13} \sqrt{\Delta m_A^2} \sqrt{1 - \sin^2 2\theta_{12} \sin^2 \alpha}. \quad (24)$$

We have that $\sin^2 2\theta_{12} \simeq \frac{1}{8}(5 + \sqrt{5}) + \epsilon_{12}^\ell \sqrt{\frac{1}{2}} \sqrt{5 - \sqrt{5}}$, and the lower limit on $|(m_\nu)_{ee}|$ is $\cos^2 \theta_{13} \sqrt{\Delta m_A^2} \cos 2\theta_{12}$, where $\cos 2\theta_{12} \simeq \frac{1}{4}(\sqrt{5} - 1) - \epsilon_{12}^\ell \sqrt{\frac{1}{2}} \sqrt{5 + \sqrt{5}}$.

As already mentioned, it is not surprising that small numbers such as θ_{23}^q and θ_{13}^q can be very well described by π/n , where $n \gg 1$. It is more the fact that the Cabibbo angle and the large mixing angles in the lepton sector can also be written as π/n , which is both interesting and for our purposes very helpful. If one insists on the π/n behavior, then the magnitude of θ_{13}^ℓ can only be speculated upon. For instance, $\sin^2 \theta_{13}^\ell = 0.016$ (0.005) corresponds to $\theta_{13}^\ell = \pi/25$ ($\pi/45$).

In summary, we have proposed a unified parametrization of the CKM and PMNS matrices by interpreting the leading mixing angles as being at zeroth order a fraction of π with an integer number. While for atmospheric neutrino mixing this is trivial, $\theta_{23}^\ell = \pi/4$, we have chosen here $\theta_{12}^\ell = \pi/5$, which is consistent with the currently allowed 2σ range, and $\theta_{12}^q = \pi/12$. The resulting sines and cosines (which correspond to physical quantities) are all rather simple, irrational but algebraic numbers. We note that $\cos \pi/5 = \varphi/2$, i.e., solar neutrino mixing is here connected to the golden ratio φ . The perturbation parameters for θ_{12}^ℓ (which are required to reproduce the central values of global fits) are of the same order as the observed deviations from tri-bimaximal mixing. They are small and for the 12-rotations of the same order and sign for both quarks and leptons.

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